Analysis of a Dynamic Voluntary Contribution Mechanism Public Good Game

23rd IPE Conference

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Why Solving the game is important?

To know optimal balance between investing in productive capacity and contributing to provision.

Examples of public goods

- Good environment
- National Defense
The Game

- 4 people in group for 10 periods
- Each period has two stages:
  1. investment stage
  2. contribution stage
- Endowments of 10 for each player in each period
Investment Stage

- Players can increase their contribution productivity from the starting value of 0.30.
- Vote (median rule) to determine the amount each player in the group will invest in increasing contribution productivity.
- Contribution productivity increases by 0.01 multiplied by the investment.

Contribution productivity \( M_t = M_{t-1} + 0.01 \cdot I_t \)

for \( t = [1..10] \)

\( M_0 = 0.3 \)
Contribution Stage

Players decide how to allocate their remaining money between private consumption and public good.

Payoff:

\[ \pi_{it} = \omega - I_t - c_{it} + M_t \sum c_{jt} \]
Example ($M_0 = 0.3$):

### Table

<table>
<thead>
<tr>
<th>Players</th>
<th>$\omega$</th>
<th>$I_t$</th>
<th>$M_t$</th>
<th>$C_{it}$</th>
<th>$M_t \sum c_{jt}$</th>
<th>$\pi_{it}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3</td>
<td>0.33</td>
<td>7</td>
<td>4.95</td>
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<td>6.95</td>
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<td></td>
<td>3</td>
<td>4.95</td>
<td>8.95</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td>11.95</td>
</tr>
</tbody>
</table>
Potential Outcomes

- **The Lowest Payoff outcome.** How would the players act to get the lowest possible payoffs? What are the lowest possible payoffs?

- **The Nash Equilibrium.** What would happen if each player acted in his own interest?

- **The Socially Optimal outcome.** How should the players act so that the sum of payoffs is maximized? What is this sum of payoffs?
Lowest payoff outcome

- Lowest possible payoff is 0
- Occurs if the group invests everything in every period and never contributes anything

Payoffs are 0 in every period and 0 at the end of 10 periods.

\[ \pi_{it} = \omega - I_t - c_{it} + M_t \sum c_{jt} \]
Nash equilibrium

- Think of the last period
- Player maximizes his payoff. If he contributes anything he reduces his payoff. Decides not to contribute.
- All players follow the same strategy
- If nobody contributes, then nobody invests
Nash equilibrium

- Everyone is left with his endowment
- Occurs for all previous periods up to the first one
- All players follow the same strategy
- Nash equilibrium is for everyone to keep his money

Each person’s payoff is $10 \cdot 10 = 100$. 
The mathematical model

\[ f(I, C) = [\omega - C - I] + [4 \cdot M_t \cdot C] \]

\[ M_t = M_{t-1} \cdot (1 + 0.01 \cdot I) \]

\[ M_0 = 0.3 \]

4 \cdot M_t \cdot C is payoff and \( \omega - C - I \) is the amount left after both stages.
Assumption

**Assumption:** the optimal result requires contributing all that is left after the investment. We can eliminate one of the two variables - $C$ or $I$.

Now $I = p \cdot \omega$ and $C = (1 - p) \cdot \omega$.

\[
\begin{align*}
    f(p) &= 4 \cdot M_t \cdot \omega \cdot (1 - p) \\
    M_t &= M_{t-1} \cdot (1 + 0.01 \cdot \omega \cdot p) \\
    M_0 &= 0.3
\end{align*}
\]

where:

- $p$ is the *proportion* of investment
- $\omega$ is the endowment (10)
- $M_t$ is the $t_{th}$ multiplier
Final model

From now, let us solve it specifically for our case, when endowment is 10.

\[ f(p) = 40 \cdot M_t \cdot (1 - p) \]

\[ M_t = M_{t-1} \cdot \left(1 + \frac{p}{10}\right) \]

\[ M_0 = 0.3 \]
Approximation

- Ran the simulation for 10 periods with step 0.1
- Time complexity of the algorithm would be $O(n^a)$

**Assumption:** The optimal solution requires that players first only invest then only contribute.
Graphical representation of computational result

Plotted by https://plot.ly/plot
Regression analysis result

\[ f(x) = 400 \cdot \left[ -m \cdot \omega \cdot x^2 + (m \cdot \omega \cdot T - M_0) \cdot x + M_0 \cdot T \right] \]

\[ x_{\text{max}} = \frac{T}{2} - \frac{M_0}{2 \cdot m \cdot \omega} \]

\[ f_{\text{max}} = f(x_{\text{max}}) = f \left( \frac{T}{2} - \frac{M_0}{2 \cdot m \cdot \omega} \right) \]

where:

- \( m \) is the increase in contribution productivity (0.01)
- \( T \) is the number of periods
- \( x \) is the stage when players switch to contributing. The number before the decimal point defines a period. The number after the decimal point defines an investment in that period.
In our specific case

\[ f(x) = -0.1x^2 + 0.7x + 3 \]

\[ x_{\text{optimal}} = 3.5 \]

which indicates investment until the 4\text{th} period and in that period investment of 5

\[ f_{\text{optimal}} = f(x_{\text{optimal}}) = 169 \]

which implies the payoff of 169.
Thank you!

Questions?
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